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Rapid Note

Time-dependent harmonic oscillator and spectral determinant on graphs

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Abstract. Using a path integral approach and also considerations about the time-dependent harmonic oscillator, we compute the spectral determinant of the operator $(-\Delta + V(x))$ on a graph. (Δ is the Laplacian and $V(x)$ is some potential defined on the graph). We recover a recent result that was obtained by constructing the Green's function on the graph. We also extend those considerations to the case when i) a magnetic field is added to the system, ii) the potential, $V(x)$, contains repulsive δ peaks.

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Spectral properties of the Laplacian operator on graphs have been investigated in several different contexts by physicists – let us simply mention superconducting networks [1], vibrational properties of fractal structures [2], weakly disordered systems [3,4], quantum chaos [5] – as well as by mathematicians [6]. During last year [4], a compact form for the determinant of the operator $(-\Delta + \gamma)$ has been obtained (γ is a constant). More recently [7], this result has been generalized to the operator $(H + \gamma)$, with $H = -\Delta + V(x)$. $V(x)$ is some potential defined in each point x of the graph. Both approaches relied on the construction of the Green's function on the graph. In this letter, we propose to recover this last result [7] by using a quite different way based on Path Integrals [8] and Time-Dependent Harmonic Oscillator (TDHO) properties [9].

To introduce those results, we consider a graph $\mathcal{G}$ made of V vertices linked by B bonds. Let us define, on each bond $(\alpha\beta)$, of length $l_{\alpha\beta}$, the coordinate $x_{\alpha\beta}$ that runs from 0 (vertex α) to $l_{\alpha\beta}$ (vertex β). (Conversely, $x_{\beta\alpha} \equiv l_{\alpha\beta} - x_{\alpha\beta}$.) To avoid cumbersome notations, ϕ being some function defined on the graph, we will simply write, when it is not ambiguous, $\phi(\alpha)$ for $\phi(x_{\alpha\beta=0})$ and $\int_\alpha^\beta \phi$ for $\int_0^{l_{\alpha\beta}} \phi(x_{\alpha\beta})\mathrm{d}x_{\alpha\beta}$.

The spectrum of H is determined by imposing continuity of the eigenfunctions φ and current conservation at each vertex α, namely:

$$\sum_{i=1}^{m_\alpha} \left.\frac{\mathrm{d}\varphi(x_{\alpha\beta_i})}{\mathrm{d}x_{\alpha\beta_i}}\right|_{x_{\alpha\beta_i}=0} = 0 \quad (1)$$

(the summation is taken over the m_α nearest vertices of α).

In the sequel, we will consider, for each bond, two independent solutions, $\psi_{\alpha\beta}$ and $\psi_{\beta\alpha}$, of the equation

$$(H + \gamma)\,\psi = 0. \quad (2)$$

Those functions are chosen to satisfy:

$$\psi_{\alpha\beta}(\alpha) = 1; \quad \psi_{\alpha\beta}(\beta) = 0 \quad (3)$$
$$\psi_{\beta\alpha}(\alpha) = 0; \quad \psi_{\beta\alpha}(\beta) = 1 \quad (4)$$

with the wronskian:

$$W_{\alpha\beta} \equiv \psi_{\alpha\beta}\frac{\mathrm{d}\psi_{\beta\alpha}}{\mathrm{d}x_{\alpha\beta}} - \psi_{\beta\alpha}\frac{\mathrm{d}\psi_{\alpha\beta}}{\mathrm{d}x_{\alpha\beta}} = \frac{\mathrm{d}\psi_{\beta\alpha}}{\mathrm{d}x_{\alpha\beta}}(\alpha)$$
$$= -\frac{\mathrm{d}\psi_{\alpha\beta}}{\mathrm{d}x_{\alpha\beta}}(\beta) = \frac{\mathrm{d}\psi_{\alpha\beta}}{\mathrm{d}x_{\beta\alpha}}(\beta). \quad (5)$$

Let us first recall the result of [4]. The authors established that:

$$\det(-\Delta + \gamma) = \gamma^{\frac{V-B}{2}} \prod_{(\alpha\beta)} \sinh(\sqrt{(\gamma)}\,l_{\alpha\beta})\ \det(M^0) \quad (6)$$

where M^0 is a $(V \times V)$ matrix with the elements:

$$M^0_{\alpha\alpha} = \sum_{i=1}^{m_\alpha} \coth(\sqrt{\gamma}\,l_{\alpha\beta_i}) \quad (7)$$
$$M^0_{\alpha\beta} = -\frac{1}{\sinh(\sqrt{\gamma}\,l_{\alpha\beta})} \quad \text{if } (\alpha\beta) \text{ is a bond}$$
$$= 0 \quad \text{otherwise} \quad (8)$$

(the summation is taken over the m_α nearest vertices of α).

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In fact, the authors also considered the case when a magnetic field is added to the system. This point will be discussed at the end of the paper.

In [7], we showed that:

$$\det(H+\gamma) \equiv \det(-\Delta + V(x) + \gamma) = \prod_{(\alpha\beta)} \frac{1}{\frac{\mathrm{d}\psi_{\beta\alpha}}{\mathrm{d}x_{\alpha\beta}}(\alpha)} \det(M) \quad (9)$$

with the $(V \times V)$ matrix M:

$$M_{\alpha\alpha} = \sum_{i=1}^{m_\alpha} \frac{\mathrm{d}\psi_{\alpha\beta_i}}{\mathrm{d}x_{\alpha\beta_i}}(\alpha) \quad (10)$$

$$M_{\alpha\beta} = M_{\beta\alpha} = \frac{\mathrm{d}\psi_{\beta\alpha}}{\mathrm{d}x_{\alpha\beta}}(\alpha) = W_{\alpha\beta} \quad \text{if } (\alpha\beta) \text{ is a bond}$$
$$= 0 \quad \text{otherwise} \quad (11)$$

(all the ψ functions appearing in (9-11) satisfy (2-4)).

When $V(x) \equiv 0$, (9-11) narrow down to (6-8). (This is actually true, up to an irrelevant multiplicative constant).

(9), like (6), was established by computing the Green's function $G(x,y)$ on the graph and using the relationship:

$$\int_{\text{Graph}} G(x,x)\mathrm{d}x = \partial_\gamma \ln \det(H+\gamma). \quad (12)$$

In the present work, we will follow quite another way essentially based on a path integral formulation [8] of the spectral determinant $S(\gamma)$. In this formalism, $S(\gamma)^{-1}$ writes:

$$S(\gamma)^{-1} \equiv \det(H+\gamma)^{-1} = \int_{\phi \text{ on graph}} \mathcal{D}\phi\mathcal{D}\bar{\phi}\mathrm{e}^{-\frac{1}{2}\int_{\text{graph}} \bar{\phi}(H+\gamma)\phi} \quad (13)$$

where ϕ is a complex field defined on the graph $\mathcal{G}$. The path integral is performed over all the fields satisfying the boundary conditions (1). General properties of path integrals lead to ($\mathrm{d}_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}$):

$$S(\gamma)^{-1} = \int \prod_{\text{vertices } \alpha} \mathrm{d}\phi_\alpha \mathrm{d}\bar{\phi}_\alpha \times \prod_{\substack{\text{bonds}\\(\alpha\beta)}} \int_{\phi(0)=\phi_\alpha}^{\phi(l_{\alpha\beta})=\phi_\beta} \mathcal{D}\phi\mathcal{D}\bar{\phi}\, \mathrm{e}^{-\frac{1}{2}\int_0^{l_{\alpha\beta}} \mathrm{d}x\, \bar{\phi}(x)(-\mathrm{d}_x^2 + V(x)+\gamma)\phi(x)} \quad (14)$$

where $\mathrm{d}\phi\mathrm{d}\bar{\phi} = \mathrm{d}\,\mathrm{Re}\,\phi\,\mathrm{d}\,\mathrm{Im}\,\phi$. This involves, after an integration by parts, the following quantity

$$\prod_{(\alpha\beta)} \int_{\phi_\alpha}^{\phi_\beta} \mathcal{D}\phi\mathcal{D}\bar{\phi}\, \mathrm{e}^{\frac{1}{2}\bar{\phi}\mathrm{d}_x\phi\big|_0^{l_{\alpha\beta}}} \mathrm{e}^{-\frac{1}{2}\int_0^{l_{\alpha\beta}} \mathrm{d}x\, (|\mathrm{d}_x\phi|^2 + (V(x)+\gamma)|\phi|^2)}$$
$$= \exp\Big(-\frac{1}{2}\sum_{\alpha=1}^{V} \bar{\phi}_\alpha \sum_{i=1}^{m_\alpha} \mathrm{d}_{x_{\alpha\beta_i}} \phi(x_{\alpha\beta_i}=0)\Big) \prod_{(\alpha\beta)} \times \int_{\phi_\alpha}^{\phi_\beta} \mathcal{D}\phi\mathcal{D}\bar{\phi}\, \mathrm{e}^{-\frac{1}{2}\int_0^{l_{\alpha\beta}} \mathrm{d}x\, (|\mathrm{d}_x\phi|^2 + (V(x)+\gamma)|\phi|^2)}. \quad (15)$$

With equation (1), the boundary terms vanish and we are left with a product of propagators of two-dimensional Time-Dependent Harmonic Oscillators. On the bond $(\alpha\beta)$, the coordinate $x_{\alpha\beta}$ is the "time" and the frequency is $\omega(x_{\alpha\beta}) = \sqrt{V(x_{\alpha\beta})+\gamma}$. Let us recall the expression of this propagator in standard notations [9]:

$$K(\mathbf{r}_b, t_b; \mathbf{r}_a; t_a) \equiv \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathcal{D}\mathbf{r}(\tau)\mathrm{e}^{-\frac{1}{2}\int_{t_a}^{t_b}(\dot{\mathbf{r}}(\tau)^2 + \omega(\tau)^2 r^2)\mathrm{d}\tau} \quad (16)$$
$$= \frac{C}{2\pi} \exp\left(\frac{1}{2}(Ar_a^2 + Br_b^2 + 2C\mathbf{r}_a \cdot \mathbf{r}_b)\right) \quad (17)$$

with ($s_a \equiv s(t_a), \dot{s}_a \equiv \frac{\mathrm{d}}{\mathrm{d}t}s(t=t_a), \ldots$):

$$A = \frac{\dot{s}_a}{s_a} + \dot{f}_a \coth(f_a - f_b) \quad (18)$$

$$B = -\frac{\dot{s}_b}{s_b} + \dot{f}_b \coth(f_a - f_b) \quad (19)$$

$$C = \sqrt{\frac{\dot{f}_a \dot{f}_b}{\sinh^2(f_a - f_b)}}\,. \quad (20)$$

The functions $s(t)$ and $f(t)$ obey the differential equations:

$$\ddot{s} + \frac{c^2}{s^3} - \omega(t)^2 s = 0 \quad (21)$$

$$\dot{f}s^2 = c \quad (22)$$

where c is an arbitrary (nonzero) constant. From the above equations, it is easy to show that the functions $\psi_\pm(t) = s(t)\mathrm{e}^{\pm f(t)}$ are two independent solutions of the equation:

$$\ddot{\psi} - \omega(t)^2\psi = 0. \quad (23)$$

Introducing two other solutions $\psi_{1,2}$ of (23) that satisfy the conditions:

$$\psi_1(t_a) = 1; \quad \psi_1(t_b) = 0 \quad (24)$$
$$\psi_2(t_a) = 0; \quad \psi_2(t_b) = 1 \quad (25)$$

and expressing $\psi_{1,2}$ in terms of $\psi\pm$, we get for the constants A, B, C in (17) the following simple form:

$$A = \dot{\psi}_1(t_a) \quad (26)$$
$$B = -\dot{\psi}_2(t_b) \quad (27)$$
$$C = \dot{\psi}_2(t_a) = -\dot{\psi}_1(t_b). \quad (28)$$

Moreover, it can be established (for instance, step by step; the proof is not difficult but rather lengthy) that, if $\omega(t)$ is real, $\psi_1(t)$ ($\psi_2(t)$) are monotonic deceasing (increasing) functions for $t_a \le t \le t_b$ and also that $\dot{\psi}_2(t_a)\dot{\psi}_1(t_b) - \dot{\psi}_1(t_a)\dot{\psi}_2(t_b) > 0$.

Now, it is a simple matter to come back to our computation of $S(\gamma)$. Considering for each bond $(\alpha\beta)$

the functions $\psi_{\alpha\beta}$ and $\psi_{\beta\alpha}$ defined in equations (2-4), we can express (15) as:

$$\prod_{(\alpha\beta)} \frac{1}{2\pi} \frac{\mathrm{d}\psi_{\beta\alpha}}{\mathrm{d}x_{\alpha\beta}}(\alpha) \;\exp\left(\frac{1}{2}\sum_{\alpha,\beta} \bar{\phi}_\alpha\, M_{\alpha\beta}\, \phi_\beta\right) \tag{29}$$

with the matrix M defined in (10, 11). Finally, Gaussian integration leads to (9) – up to an inessential normalization constant.

Adding a magnetic field [4,8], we must replace all the derivatives appearing in the Laplacian and also in the current conservation condition (1) by covariant derivatives $\mathrm{D}_x = \mathrm{d}_x - \mathrm{i}A(x)$ ($A(x)$ is the vector potential). For instance, (1) becomes

$$\sum_{i=1}^{m_\alpha} \mathrm{D}_{x_{\alpha\beta_i}} \varphi(x_{\alpha\beta_i} = 0) = 0 \tag{30}$$

Computing the spectral determinant along the same lines (13 -15) as before we get, for (15), a product of terms of the form

$$\int_{\phi_\alpha}^{\phi_\beta} \mathcal{D}\phi\mathcal{D}\bar{\phi}\; \mathrm{e}^{-\frac{1}{2}\int_\alpha^\beta (|\mathrm{D}_x\phi|^2 + (V(x)+\gamma)|\phi|^2)} \tag{31}$$

(in the integration by parts, the boundary terms still vanish because of (30)). For each bond $(\alpha\beta)$ one may recover the propagator of a TDHO by performing the following gauge transformation

$$\phi(x) = \tilde{\phi}(x)\mathrm{e}^{\mathrm{i}\int_{x_0}^{x} A(x')} \tag{32}$$

where the integral is performed along $(\alpha\beta)$ and x_0 is an arbitrary point on this bond.

Let us define

$$\theta_{\beta\alpha} = \int_\alpha^\beta A(x) \tag{33}$$

($\theta_{\beta\alpha} = -\theta_{\alpha\beta}$) and choose x_0 such that

$$\int_{x_0}^{\beta} A(x) = -\int_{x_0}^{\alpha} A(x) = \frac{1}{2}\theta_{\beta\alpha}.$$

Then, (31) becomes

$$\int_{\phi_\alpha \mathrm{e}^{-\mathrm{i}\theta_{\alpha\beta}/2}}^{\phi_\beta \mathrm{e}^{-\mathrm{i}\theta_{\beta\alpha}/2}} \mathcal{D}\tilde{\phi}\mathcal{D}\bar{\tilde{\phi}}\; \mathrm{e}^{-\frac{1}{2}\int_\alpha^\beta (|\mathrm{d}_x\tilde{\phi}|^2 + (V(x)+\gamma)|\tilde{\phi}|^2)}.$$

Considering the bounds of this integral, the quantity $\bar{\phi}_\alpha M_{\alpha\beta}\phi_\beta$ appearing in (29) must be changed into

$$\bar{\phi}_\alpha \mathrm{e}^{+\mathrm{i}\theta_{\alpha\beta}/2}\, M_{\alpha\beta}\, \phi_\beta \mathrm{e}^{-\mathrm{i}\theta_{\beta\alpha}/2}. \tag{34}$$

The final result is that (9) still holds provided the off-diagonal elements of the matrix M in (11) are slightly modified:

$$M_{\alpha\beta} \longrightarrow M_{\alpha\beta}\, \mathrm{e}^{+\mathrm{i}\theta_{\alpha\beta}}. \tag{35}$$

Such a modification for the matrix M^0 was already obtained in [4].

Let us now study what happens when we replace current conservation (1) by generalized boundary conditions

$$\sum_{i=1}^{m_\alpha} \mathrm{d}_{x_{\alpha\beta_i}} \varphi(x_{\alpha\beta_i} = 0) = \lambda_\alpha \varphi(\alpha). \tag{36}$$

The boundary contribution in (15) will not vanish but rather produce an additional term $\exp(-\sum_{\alpha=1}^{V} \lambda_\alpha |\phi_\alpha|^2/2)$. Thus, (9) is still correct if, this time, we change the diagonal elements (10) of M:

$$M_{\alpha\alpha} \longrightarrow M_{\alpha\alpha} - \lambda_\alpha. \tag{37}$$

Finally, let us discuss the case when the potential contains repulsive δ peaks. In fact, it is enough to consider only one such peak located at some point c on the link (ab): $V(x) \equiv V_1(x) + \lambda_c\delta(x-c)$ where $V_1(x)$ is regular ($H \equiv H_1 + \lambda_c\delta(x-c)$). The generalization to several peaks is straightforward.

The path integral for the bond (ab) becomes:

$$\int_{\phi_a}^{\phi_b} \mathcal{D}\phi\mathcal{D}\bar{\phi}\; \mathrm{e}^{-\frac{1}{2}\int_a^b \bar{\phi}(H+\gamma)\phi}$$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\mathcal{D}\bar{\phi}\; \mathrm{e}^{-\frac{1}{2}\int_a^b \bar{\phi}(H_1+\gamma)\phi} \mathrm{e}^{-\frac{1}{2}\int_a^b \bar{\phi}\lambda_c\delta(x-c)\phi} \tag{38}$$

$$= \int \mathrm{d}\phi_c \mathrm{d}\bar{\phi}_c \mathrm{e}^{-\frac{1}{2}\lambda_c|\phi_c|^2} \int_{\phi_a}^{\phi_c} \mathcal{D}\phi\mathcal{D}\bar{\phi}\; \mathrm{e}^{-\frac{1}{2}\int_a^c \bar{\phi}(H_1+\gamma)\phi}$$

$$\times \int_{\phi_c}^{\phi_b} \mathcal{D}\phi\mathcal{D}\bar{\phi}\; \mathrm{e}^{-\frac{1}{2}\int_c^b \bar{\phi}(H_1+\gamma)\phi} \tag{39}$$

We conclude that we must compute the spectral determinant on the new graph consisting in $(V+1)$ vertices (including c) and $(B+1)$ bonds ((ab) is replaced by (ac) and (cb)). Moreover, in c, we have to consider generalized boundary conditions.

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